CMPT260.3-01 Final Exam
December 9, 2006
Three (3) hours.
Closed book. No calculators or computers.

Instructions: Answer all questions in a University of Saskatchewan examination booklet. A portion of the marks awarded will be for the style and clarity of your answer. Do not use a hard pencil. There are 100 marks on this exam, the values appear in bold next to the questions.

If you are a student who has been excused from an assignment/quiz/midterm during the term, please write a note to this effect on the cover of your exam booklet.

1. Logic
   a. **(4)** Give truth tables for $P \vee Q$ and $P \Rightarrow Q$.
   b. **(4)** Each of four cards in a deck has a single character printed on each side. A card dealer tells you the following about this deck. "*If* a card has a vowel on one side, *then* it has an even number on the other side". The four cards are dealt out and you see the following characters:
   A B 8 5
   How many cards (and which ones) must you turn over to verify the truth of the dealer's statement?
   c. **(6)** Give the truth table for
   $$(C \vee \neg B) \Leftrightarrow (B \Rightarrow (B \wedge C)).$$
   d. **(4)** Prove
   $$(A \Leftrightarrow (A \wedge B)) \vdash A \Rightarrow B$$
   e. **(4)** Prove
   $$A \Rightarrow C,\ B \Rightarrow \neg C,\ A \vdash \neg B$$
   f. **(4)** Simplify the following expression by moving all negations in:
   $$\neg(P \vee (Q \wedge \neg P) \vee \neg(\neg Q \wedge P)).$$

2. Induction
   a) **(4)** Give the definition of addition.

   b) **(6)** Using only the Peano postulates, and the definition of addition, prove the associative property of addition:
   $$(a + b) + n = a + (b + n).$$

   c) **(6)** The *height* of a single node is 0. The height of a binary tree is the maximum of the heights of its two subtrees, plus 1. Show, by induction, that the number of nodes in a complete (no nodes missing) binary tree of height n is $2^{n+1} - 1$.

$\frac{4}{6}$

DO EITHER QUESTION 3 OR QUESTION 4.

3. **(10)** More induction
   a. Define exponentiation.
   b. Assume that the following two theorems have been proven:
      Theorem 9: $(ab)^x = a^x b^x$.
      Theorem 10: $a^n a^m = a^{n+m}$.

      Using only these two theorems about exponentiation, and familiar facts about addition and multiplication, prove
      $(a^n)^m = a^{nm}$.

4. (10) Prolog

Define predicates `rotatel( L1, L2 ),` and `rotater( L1, L2 )` such that *L2* is the result of rotating all elements of a list *L1* one unit to the left or to the right, respectively. For example:

```
?- rotatel( [], L)
L = []
?- rotatel( [a], L).
L = [a]
?- rotatel( [a,b,c], L).
L = [b,c,a]
?- rotater( [a,b,c,d], L).
L = [d,a,b,c]
```

Your solution should not call any other predicates, and should return no other answers on backtracking.

5. Functions
   a. **(4)** Show that the function $F: R \rightarrow R$ given by $f(x) = x^3 - 8$ is a bijection, using the definitions of 1-1 and onto.
   b. **(6)** Show that the function $F: R \rightarrow R$ given by $f(x) = x^2 - 8$ is not a bijection.
   c. **(6)** For any natural number k, find a bijection between the natural numbers (0, 1, 2, 3 …) and the set (k, k+1, k+2, k+3, …). Prove your answer.

6. Sets and Relations
    a. **(6)** What is the powerset of the set $A = \{a,b,c\}$? If $B = \{0,1\}$, what is $A \times B$?
    b. **(8)** Let $S = \{r,s,t,u\}$. Let $R = \{(s,t),(u,r)\}$. Find
        i. The reflexive closure of R,
        ii. The symmetric closure of R,
        iii. The transitive closure of R,
        iv. The reflexive, symmetric, and transitive closure of R.
    c. **(4)** Define *equivalence relation.*
    d. **(6)** Define *set equality*. Prove that set equality is an equivalence relation.

7. **(8)**

a. Define what is meant by the *natural join* of two relations.

b. Let A and B be the relations given by the following tables:

A	*Character*	*Handle*	*MembershipDate*
	Elmer	123	25-04-92
	Bugs	118	19-06-94
	Tweety	617	11-08-96
	Sylvester	622	08-09-96
	Pepe	417	02-11-94

B	*Handle*	*Rank*
	417	top
	622	second
	123	second

Give

a) The restriction of A to persons who joined *strictly before* 1996. (The last two digits of the *MembershipDate* gives the date of joining.) Call this new relation C.

b) The projection of C onto attributes *Character* and *Handle*. Call the new relation D.

c) The natural join of D and B.

End of Exam

Equivalences, Propositional Calculus

Law	*Name*
$P \vee \neg P \equiv \mathrm{T}$ $P \wedge \neg P \equiv \mathrm{F}$	Excluded middle law Contradiction law
$P \vee \mathrm{F} \equiv P$ $P \wedge \mathrm{T} \equiv P$	Identity laws
$P \vee \mathrm{T} \equiv \mathrm{T}$ $P \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent laws
$\neg(\neg P) \equiv P$	Double negation law
$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative laws
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative laws
$(P \vee Q) \wedge (P \vee R) \equiv P \vee (Q \wedge R)$ $(P \wedge Q) \vee (P \wedge R) \equiv P \wedge (Q \vee R)$	Distributive laws
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's laws

Main rules of inference

$A,\ B \models A \wedge B$	Law of Combination
$A \wedge B \models B$	Law of Simplification
$A \wedge B \models A$	Variant of Law of Simplification
$A \models A \vee B$	Law of Addition
$B \models A \vee B$	Variant of Law of Addition
$A,\ A \Rightarrow B \models B$	Modus Ponens
$\neg B,\ A \Rightarrow B \models \neg A$	Modus Tollens
$A \Rightarrow B,\ B \Rightarrow C \models A \Rightarrow C$	Hypothetical Syllogism
$A \vee B,\ \neg A \models B$	Disjunctive Syllogism
$A \vee B,\ \neg B \models A$	Variant of Disjunctive Syllogism
$A \Rightarrow B,\ \neg A \Rightarrow B \models B$	Law of Cases
$A \Leftrightarrow B \models A \Rightarrow B$	Equivalence Elimination
$A \Leftrightarrow B \models B \Rightarrow A$	Variant of Equivalence Elimination
$A \Rightarrow B,\ B \Rightarrow A \models A \Leftrightarrow B$	Equivalence Introduction
$A, \neg A \models B$	Inconsistency Law
$B \Rightarrow A \wedge \neg A \models \neg B$	Indirect Proof

P->Q = !P V Q

P<->Q = (P ∧ Q) V (!P ∧ !Q)

P<->Q = (P->Q) ∧ (Q->P)